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Regular quantizations of Kähler manifolds and constant scalar curvature metrics

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Abstract

In this paper we prove that if a Kähler manifold (M, ω) admits a regular quantization then its scalar curvature is constant. Moreover, we apply this result to the two-dimensional complete Reinhardt domains in $\mathbb{C}^2$ to show that such domains admit a regular quantization iff they are biholomorphically isometric to the 2-ball in $\mathbb{C}^2$ endowed with the hyperbolic metric.
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1. Introduction

In the quantum mechanic terminology introduced by Kostant and Souriau, a geometric quantization of a Kähler manifold (M, ω) is a pair (L, h), where L is a holomorphic line bundle over M, called the *quantum line bundle*, and h is an Hermitian metric on L, such that $\mathrm{Ric}(h) = \omega$. Here $\mathrm{Ric}(h)$ is the 2-form on M defined by the equation:

$$\mathrm{Ric}(h) = -\frac{i}{2\pi}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)), \tag{1}$$

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for a trivializing holomorphic section $\sigma : U \subset M \to L \setminus \{0\}$ of L. A Kähler manifold (M, ω) admits a geometric quantization iff ω represents the first Chern class of the line bundle L and this happens iff ω is an integral form. For any non-negative integer m, consider L^m the mth tensor power of L. The Hermitian metric h induces, in a natural way, an Hermitian metric h_m on L^m, such that $\mathrm{Ric}(h_m) = m\omega$. Therefore, (L^m, h_m) is a geometric quantization of the Kähler manifold (M, $m\omega$). In this context one can define a smooth function $\epsilon_{m\omega}$, on M, called the *epsilon* function (see Section 3 below) which is one of the key ingredients in the framework of quantization of Kähler manifolds carried out in [3,5–8].

A geometric quantization (L, h) of the Kähler manifold (M, ω) is said to be *regular* if the function $\epsilon_{m\omega}$ is a non-zero (hence positive) constant for any non-negative integer m.

In [5,6] Cahen, Gutt and Rawnsley, obtain a deformation quantization of (M, ω) by generalizing Berezin's method [3] to the case of compact Kähler manifolds which admit a regular quantization. Compact Kähler manifold admitting a regular quantization play also a fundamental role for the stability of Kähler–Einstein metric [16], the stability of holomorphic line bundle and the existence of a constant scalar curvature metric in a fixed cohomology class (see [10,1,2]).

The aim of this article is to investigate the geometric properties of regular quantizations for *non-compact* Kähler manifolds.

Our main results are Theorems 4.1 and 4.7. In Theorem 4.1 we prove that *a Kähler manifold admitting a regular quantization has constant scalar curvature metric*. Its proof is based on the asymptotic expansion of a Laplace integral on a real analytic Kähler manifold due to Engliš [13] (see Lemma 2.2 below) and on the properties of the epsilon function developed by Cahen, Gutt and Rawnsley in [6] (see Lemma 3.1 below). In Theorem 4.7, by applying Theorem 4.1, we prove that *a complete Reinhardt domain in* $\mathbb{C}^2$ *admits a regular quantization iff it is biholomorphically isometric to the hyperbolic 2-space.*

The paper is organized as follows. In Section 2, we recall the above mentioned result of Engliš about the Laplace integral. The geometric quantization tools needed for the proof of our main results are collected in Section 3. Section 4 is dedicated to the proof of Theorems 4.1 and 4.7.

2. A Laplace integral on a Kähler manifold

We refer to [12] for the material of this section and for further results. Let M be a n-dimensional complex manifold endowed with a real analytic Kähler metric g and let ω be the corresponding Kähler form. Let Φ be a Kähler potential for the metric g, namely a real valued function Φ defined on a open set $U \subset M$ satisfying

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\Phi. \tag{2}$$

If $g = \sum_{j\bar{k}}^{n} g_{j\bar{k}} \mathrm{d}z_j \mathrm{d}\bar{z}_k$ is the local expression of the metric g then the previous equation is equivalent to

$$g_{j\bar{k}} = \frac{1}{\pi}\frac{\partial^2\Phi}{\partial z_j \partial\bar{z}_k}. \tag{3}$$

The potential Φ can be complex analytically continued to an open neighbourhood $W \subset U \times U$ of the diagonal. Denote this extension by $\Phi(x, \bar{y})$. It is holomorphic in x and anti-holomorphic in y and, with this notation, $\Phi(x) = \Phi(x, \bar{x})$. Observe also that $\overline{\Phi(x, \bar{y})} = \Phi(\bar{x}, y)$. Consider the real valued function

$$D(x, y) = \Phi(x, \bar{x}) + \Phi(y, \bar{y}) - \Phi(x, \bar{y}) - \Phi(y, \bar{x})$$

on W. It is easily seen that the function $D(x, y)$ is independent from the potential chosen which is defined up to the sum with the real part of a holomorphic function. Calabi [9] christened the function $D(x, y)$ the *diastasis function*. We refer to [9] for details and further results on the diastasis function.

For all $x \in U$ (U as above), the positive definiteness of the matrix (3) implies that the function

$$D(x, \cdot) = \Phi(x, \bar{x}) + \Phi(\cdot, \bar{\cdot}) - \Phi(x, \bar{\cdot}) - \Phi(\cdot, \bar{x})$$

has a local minimum at x. Shrinking U, if necessary, we can assume that $D(x, y)$ is a globally defined on $U \times U$, $D(x, y) \geq 0$ and $D(x, y) = 0$ iff $x = y$.

Example 2.1. Consider the complex projective space $\mathbb{C}P^N$, $N \leq +\infty$ endowed with the Fubini–Study form Ω_{FS}. The diastasis can be written in terms of the coordinates in $\mathbb{C}^{N+1}$ as

$$D_{FS}(\pi(z), \pi(w)) = \log \frac{\|z\|^2 \|w\|^2}{|\langle z, w\rangle|^2},$$

where $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{C}P^N$ is the canonical projection and where we are denoting by $\langle \cdot, \cdot \rangle$ the standard Hermitian metric on $\mathbb{C}^{N+1}$. In particular $D > 0$ unless $\pi(z) = \pi(w)$ where $D = 0$. Observe also that the function $\mathrm{e}^{-D_{FS}(\pi(z),\pi(w))} = |\langle z, w\rangle|^2 / \|z\|^2 \|w\|^2$ is globally defined on $\mathbb{C}P^N \times \mathbb{C}P^N$ and is equal to one on the diagonal.

Let α be a positive real number and let $U \subset M$ as above. Consider the integral

$$L_\alpha(x) = \int_U \mathrm{e}^{-\alpha D(x,y)} \frac{\omega^n}{n!}(y), \tag{4}$$

Before stating Engliš' main result about this integral (Theorem 2.2 below), we fix our notations and conventions.

The curvature tensor is defined as

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}, \; i, j, k, l = 1, \ldots, n$$

The Ricci curvature is

$$\mathrm{Ric}_{i\bar{j}} = -\sum_{k,l=1}^{n} g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} (\log det g_{j\bar{k}}), \; i, j = 1, \ldots, n \tag{5}$$

and the scalar curvature is the trace of the Ricci curvature

$$scal_g = -\sum_{i,j=1}^{n} g^{i\bar{j}} \mathrm{Ric}_{i\bar{j}}. \tag{6}$$

Here $g^{i\bar{j}}$ denotes the inverse matrix of $g_{i\bar{j}}$. The Laplace operator, denoted by Δ, is given by

$$\Delta f = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.$$

Finally, we set

$$|R|^2 = \sum_{i,j,k,l=1}^{n} |R_{i\bar{j}k\bar{l}}|^2, \; |\mathrm{Ric}|^2 = \sum_{i,j=1}^{n} |\mathrm{Ric}_{i\bar{j}}|^2.$$

We are now in the position to state Engliš's result.

Lemma 2.2. *(Engliš) If the integral (4) exists for some $\alpha = \alpha_0$ then it also exists for all $\alpha > \alpha_0$ and as $\alpha \to +\infty$ it has an asymptotic expansion*

$$L_\alpha(x) \sim \left(\frac{1}{\alpha}\right)^n \sum_{r \geq 0} \alpha^{-r} c_r(x), \tag{7}$$

where c_r are smooth functions on U. In particular

$$\begin{cases} c_0(x) = 1 \\ c_1(x) = -\frac{1}{2} scal_g \\ c_2(x) = -\frac{1}{3}\Delta scal_g + \frac{1}{8} scal_g^2 + \frac{1}{6}|\mathrm{Ric}|^2 - \frac{1}{24}|R|^2, \end{cases} \tag{8}$$

Remark 2.3. Observe that the factor π^n which appears in the expansion (2.21) given by Engliš in [13] is missing in (7) above since Engliš uses the expression $\omega = i/2\partial\bar{\partial}\Phi$ which differs from (2) by the factor $1/\pi$.

3. Some properties of the epsilon function

We refer to [6] and [7] for the material of this section and further results. Let (L, h) be a geometric quantization of a Kähler manifold (M, ω). Consider the space $\mathcal{H}_m \subset H^0(L^m)$ consisting of global holomorphic sections s of L^m which are bounded with respect to

$$||s||_{h_m} = \langle s, s\rangle_{h_m} = \int_M h_m(s(x), s(x)) \frac{(m\omega)^n}{n!}.$$

From now on we suppose that $\mathcal{H}_m$ is non-empty. One can show that $\mathcal{H}_m$ is a separable complex Hilbert space. Let $x \in M$ and $q \in L^m \setminus \{0\}$ a fixed point of the fibre over x. If one evaluates $s \in \mathcal{H}_m$ at x, one gets a multiple $\delta_q^m(s)$ of q, i.e. $s(x) = \delta_q^m(s)q$. The map $\delta_q^m : \mathcal{H}_m \to \mathbb{C}$ is a continuous linear functional (cfr. [5]) hence by Riesz's theorem, there exists a unique $\mathrm{e}_q^m \in \mathcal{H}_m$ such that $\delta_q^m(s) = \langle s, \mathrm{e}_q^m \rangle_{h_m}$, $\forall s \in \mathcal{H}_m$, i.e.

$$s(x) = \langle s, e_q^m \rangle_{h_m} q. \tag{9}$$

It follows by (9) that

$$\mathrm{e}_{cq}^m = \bar{c}^{-1}\mathrm{e}_q^m, \quad \forall c \in \mathbb{C}^*.$$

The holomorphic section $\mathrm{e}_q^m \in \mathcal{H}_m$ is called the *coherent state* relative to the point q. It follows that

$$\epsilon_{m\omega}(x) = h_m(q, q)|\mathrm{e}_q^m|_{h_m}^2, \tag{10}$$

is a well-defined non-negative and smooth function on M.

In order to relate the function $\epsilon_{m\omega}$ to the Kähler geometry of (M, ω) we assume throughout all this paper that: *for m sufficiently large and for every $x \in M$ there exists $s \in \mathcal{H}_m$ such that $s(x) \neq 0$.*

The previous assumption has three important consequences.

First, it implies that $\mathrm{e}_q^m \neq 0$, for m sufficiently large and, therefore, the function $\epsilon_{m\omega}$ is strictly positive on M.

Secondly, for m sufficiently large, we can define a holomorphic map

$$\varphi_m : M \to \mathbb{C}P^N, \tag{11}$$

called the *coherent states map* as follows. Let $N + 1$ ($N \leq +\infty$) be the complex dimension of the Hilbert space $\mathcal{H}_m$ and let $(s_0, \ldots, s_N)$ be a orthonormal basis for $\mathcal{H}_m$. If the previous assumption holds then, for a trivialising holomorphic section $\sigma : U \to L$ on a open set $U \subset M$, one can define a holomorphic map

$$\varphi_\sigma : U \to \mathbb{C}^N \setminus \{0\} : x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \ldots, \frac{s_N(x)}{\sigma(x)} \right). \tag{12}$$

If $\tau : V \to L$ is another holomorphic trivialisation then there exists a non-vanishing holomorphic function f on $U \cap V$, such that $\sigma(x) = f(x)\tau(x)$. The coherent states map (11) is the map whose local expression in the open set U is given by (12).

The relation between this map and the function $\epsilon_{m\omega}$ can be read in the following formula due to Rawnsley (see [15]):

$$\varphi_m^*(\Omega_{FS}) = m\omega + \frac{i}{2\pi}\partial\bar{\partial}\log\epsilon_{m\omega}. \tag{13}$$

Therefore, $\epsilon_{m\omega}$ measures the obstruction for the Kähler form $m\omega$ to be projectively induced via the coherent states map φ_m.

In the non-compact case the previous assumption is equivalent to the absence of based points in the Kodaira's theory and the map (11) is the Kodaira map written in the orthonormal basis $s_0, \ldots, s_N$. Therefore, in the compact case our assumption is satisfied for m sufficiently large.

The third implication of our assumption above is that one can define the 2-point function

$$\psi_m(x, y) = \frac{|\langle e^m_q, e^m_{q'}\rangle_{h_m}|^2}{\|e^m_q\|^2_{h_m}\|e^m_{q'}\|^2_{h_m}}, \tag{14}$$

where q and q' are non-zero elements in the fibres of x and y, respectively (see [6] and [7] for the use of this 2-point function in the quantization context).

The proof of the following lemma can be found in [6].

Lemma 3.1. *Let (L, h) be a geometric quantization of a real analytic n-dimensional Kähler manifold (M, ω). For m sufficiently large, consider the function $\epsilon_{m\omega}$ on M and the 2-point function $\psi_m(x, y)$ on $M \times M$ given by formulae (10) and (14). Then the following holds true*:

$$e^{-mD(x,y)}|\epsilon_{m\omega}(x, \bar{y})|^2 = \epsilon_{m\omega}(x)\epsilon_{m\omega}(y)\psi_m(x, y), \tag{15}$$

$$m^n \int_M \psi_m(x, y)\epsilon_{m\omega}(y)\frac{\omega^n(y)}{n!} = 1, \tag{16}$$

where $\epsilon_{m\omega}(x, \bar{y})$ denotes the analytic extension in a neighbourhood of the diagonal of the function $\epsilon_{m\omega}(x)$.

Remark 3.2. Observe that since the right hand side of (15) is globally defined on $M \times M$ then the function $e^{-mD(x,y)}|\epsilon_{m\omega}(x, \bar{y})|^2$ is a well-defined function on $M \times M$ even if the single functions $e^{-mD(x,y)}$ and $|\epsilon_{m\omega}(x, \bar{y})|^2$ are a priori defined only in a neighbourhood of the diagonal.

4. Proof of the main results

In this section we proof Theorem 4.1, and its consequence in the one-dimensional case (see Corollary 4.4 below). We also classify the regular quantization of a two-dimensional complete Reinhardt domain (see Theorem 4.7).

Theorem 4.1. *Let (M, ω) be a Kähler manifold and let g be its associated Kähler metric. If (M, ω) admits a regular quantization then $scal_g$, the scalar curvature of g, is constant. If moreover Ric_g, the Ricci tensor of g, vanishes then the norm of the curvature tensor is constant.*

Proof. Since the quantization is regular it follows by (13) (with $m = 1$) that ω is projectively induced via the map φ_1, namely $\varphi_1^*(\Omega_{FS}) = \omega$. By a result of Calabi [9] the Kähler form ω is then real-analytic and its diastasis function D is obtained by the restriction, via the map φ, of the diastasis function D_{FS} on $\mathbb{C}P^N$, namely $\varphi^*(D_{FS}) = D$. It follows by Example 2.1 that the diastasis function $D(x, y)$ vanishes if and only if $x = y$ and moreover the function $e^{-D(x,y)}$ is globally defined on $M \times M$. This function admits the points of the diagonal as critical points. In fact at these points it has its maximum value, namely 1 and $e^{-D(x,y)} = 1$ if and only if $x = y$. From this, it follows that the expansion (7), valid in the open set $U \subset M$, can be extended to all of M, namely (with $\alpha = m$) one has:

$$m^n \int_M e^{-mD(x,y)} \frac{\omega^n(y)}{n!} = 1 - \frac{c_1(x)}{m} + \frac{c_2(x)}{m^2} + \cdots \tag{17}$$

Using again the fact that the quantization is regular, it follows that the 2-point function $\psi_m(x, y)$ can be defined on $M \times M$ and by formulae (15) and (16) we can write

$$m^n \int_M e^{-mD(x,y)} \frac{\omega^n(y)}{n!} = \epsilon_{m\omega}. \tag{18}$$

□

Finally, formulae (17) and (18) together with (8) implies that $scal_g$ is constant and if, moreover, the metric g is Ricci flat then also $|R|^2$ is constant.

Remark 4.2. Observe that in [7] one can find the proof that in the regular case there exists an asymptotic expansion for $\epsilon_{m\omega}$ as in formula (18) above. Nevertheless, here we are computing its coefficients explicitly.

Remark 4.3. In the compact case the proof of our Theorem 4.1 can be easily deduced by Tian–Yau–Zelditch asymptotic expansion (see [1]). Observe also that the hypothesis on the vanishing of the Ricci tensor in our Theorem 4.1 automatically implies that the manifold M is not compact. Indeed, by a result of Hulin [11], it cannot exist a Ricci flat and projectively induced Kähler form ω on a compact complex manifold M.

As a corollary of Theorem 4.1 one gets the following:

Corollary 4.4. *Let (M, ω) be a one-dimensional and simply-connected Kähler manifold which admits a regular quantization. Then (M, ω) is biholomorphically equivalent to a complex space form, namely either the flat space $(\mathbb{C}, \omega = \frac{1}{2} dz \wedge d\bar{z})$, the one-dimensional unitary disk in $\mathbb{C}$ endowed with the hyperbolic metric, or the projective space $\mathbb{C}P^1$ endowed with the Fubini–Study metric.*

We now study the case of two-dimensional complete Reinhardt domains. Observe that Berezin's quantization of these domains has been extensively studied in [13] to whom we refer for further results.

Recall that a domain $M \subset \mathbb{C}^2$ is called *Reinhardt* if $z = (z_1, z_2) \in M$ whenever $w = (w_1, w_2) \in M$ and $|z_1| = |w_1|, |z_2| = |w_2|$. If the same holds even for all z with $|z_1 \leq |w_1|$ and $|z_2 \leq |w_2|$, the Reinhardt domain is called *complete*. One can show that any complete Reinhardt domain is of the form

$$M = \mathcal{D}_F = \{(z_1, z_2) \in \mathbb{C}^2 || z_1|^2 < x_0, |z_2|^2 < F(|z_1|^2)\}, \tag{19}$$

where $F : [0, x_0) \to (0, +\infty]$ is a non-increasing lower semi-continous function from the interval $[0, x_0) \subset \mathbb{R}$ to the extended positive reals $(0, +\infty]$ (the case $x_0 = +\infty$ is not excluded).

In the hypothesis that $F(0) < \infty$, one can define a real 2-form on $\mathcal{D}_F$ by

$$\omega_F = \frac{i}{2} \partial\bar{\partial} \log \frac{1}{F(|z_1|^2) - |z_2|^2}. \tag{20}$$

Example 4.5. In the case $F(x) = 1 - x$, $(\mathcal{D}_F, \omega_F)$ is the open 2-ball $\mathcal{D}_2$ in $\mathbb{C}^2$ endowed with the hyperbolic Kähler form $\omega_F = \omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - |z_1|^2 - |z_2|^2)$. In this case $\mathcal{D}_2$ admits a regular quantization (see e.g. [7]).

The following Proposition gives us the conditions under which ω_F is a Kähler form on $\mathcal{D}_F$.

Proposition 4.6. *Assume that F is continous on* $[0, x_0)$ and C^2 on $(0, x_0)$. *The following conditions are equivalent*:

(i) ω_F *is a Kähler form on* $\mathcal{D}_F$.
(ii) *the function* $A(x) = -xF^{'}(x)/F(x)$, *satisfies* $A^{'}(x) > 0, \quad \forall x \in [0, x_0)$, *where* F' *denotes the first derivative of F with respect to x.*
(iii) $\mathcal{D}_F$ *is strongly pseudoconvex.*

Proof. For the proof see [12]. We just give here the proof of the equivalence $(i) \Leftrightarrow (ii)$ since we wiil need it later.

Let $\omega_F = i/2\sum_{j,k=1}^{2} g_{j\bar{k}}dz_j \wedge d\bar{z}_k$ be the expression of the Kähler form ω_F in the (global) coordinates (z_1, z_2). A simple calculation shows that

$$\begin{aligned} g_{1\bar{1}} &= \left.\frac{-HF' - HxF'' + xF'^2}{H^2}\right|_{x=|z_1|^2}, \\ g_{1\bar{2}} &= \overline{g_{2\bar{1}}} = \left.\frac{-F'}{H^2}\bar{z}_1 z_2\right|_{x=|z_1|^2}, \\ g_{2\bar{2}} &= \left.\frac{F}{H^2}\right|_{x=|z_1|^2}, \end{aligned} \tag{21}$$

where H is the real valued function on $\mathcal{D}_F$ defined by $H(z_1, z_2) = F(|z_1|^2) - |z_2|^2$. An easy calculation shows that:

$$\det g_{j\bar{k}} = g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2 = -\frac{F^2}{H^3}\left(\frac{xF'}{F}\right)'\Bigg|_{x=|z_1|^2}. \tag{22}$$

The form ω_F satisfies the Kähler condition if and only if the matrix $g_{j\bar{k}}$ is positive definite and, since $g_{2\bar{2}} > 0$, this is the case if and only if $\det g_{j\bar{k}} > 0$ which, by (22), turns out to be equivalent to (ii). □

In what follows we will suppose ω_F is a Kähler form. Since we only work in the smooth case we will also assume that F is a smooth function on $[0, x_0)$.

Observe that the Kähler form ω_F is exact and hence integral. Therefore (D_F, ω_F) admits a geometric quantization.

We are now in the position to prove the main result of this section.

Theorem 4.7. *Let* D_F *be a complete Reinhardt domain and suppose that* ω_F *is a Kähler form on* D_F. *Then* (D_F, ω_F) *admits a regular quantization iff* (D_F, ω_F) *is biholomorphically isometric to* (D_2, ω_{hyp}).

Proof. The "if" part follows by Example 4.5. The opposite implication is a consequence of our Theorem 4.1 and of the following Theorem 4.8, interesting on its own sake. □

Theorem 4.8. *Let D_F be a complete Reinhardt domain and suppose that ω_F is a Kähler form on D_F. Suppose that $scal_{g_F}$, the scalar curvature of the metric g_F, equals a constant. Then (D_F, ω_F) is biholomorphically isometric to (D_2, ω_{hyp}).*

Proof. A straightforward calculation using (21), (22) and (5) gives the following expression for the component of the Ricci tensor for the metric g_F:

$$\begin{aligned} \mathrm{Ric}_{1\bar{1}} &= -3g_{1\bar{1}} - A(|z_1|^2),\\ \mathrm{Ric}_{1\bar{2}} &= \overline{\mathrm{Ric}_{2\bar{1}}} = -3g_{1\bar{2}},\\ \mathrm{Ric}_{2\bar{2}} &= -3g_{2\bar{2}}, \end{aligned} \tag{23}$$

where $A(|z_1|^2)$ is the function defined on $D_{x_0} = \{z_1 \in \mathbb{C} | |z_1|^2 < x_0\}$ by:

$$A(|z_1|^2) = \frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}\log(xF'^2 - FF' - xFF'')\right]\bigg|_{x=|z_1|^2} \tag{24}$$

The scalar curvature is given by:

$$\begin{aligned} scal_{g_F} &= \sum_{i,j=1}^{2} g^{i\bar{j}}\mathrm{Ric}_{i\bar{j}} = -6 - g^{1\bar{1}}A(|z_1|^2) = -6 - \frac{g_{2\bar{2}}}{\det g_{j\bar{k}}}A(|z_1|^2)\\ &= -6 + \frac{H}{F(xF'/F)'|_{x=|z_1|^2}}A(|z_1|^2). \end{aligned} \tag{25}$$

Suppose that $scal_{g_F}$ equals a constant. Since $H = F(|z_1|^2) - |z|^2$ depends on $|z_2|^2$ and $A(|z_1|^2)/F(xF'/F)'\,|_{x=|z_1|^2}$ depends only on $|z_1|^2$ this forces A to be identically zero. By expression (24) we get:

$$\begin{aligned} &\frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}\log(xF'^2 - FF' - xFF'')\right]\bigg|_{x=|z_1|^2}\\ &\quad = \Delta(\log(xF'^2 - FF' - xFF'')|_{x=|z_1|^2}) = 0 \end{aligned}$$

where $\Delta = \partial/\partial x_1^2 + \partial/\partial y_1^2$ is the standard Laplacian with respect to x_1, y_1, the real part and imaginary part of z_1, respectively. This means that the function $\log(xF'^2 - FF' - xFF'')\big|_{x=|z_1|^2}$, defined on the disk D_{x_0}, is harmonic. Thus

$$xF'^2 - FF' - xFF''\Big|_{x=|z_1|^2} = |\Phi(z_1)|^2, \tag{26}$$

for some holomorphic function Φ on D_{x_0}.

Observe also that the fact that $A(|z_1|^2) = 0$ implies, by (23), that $\mathrm{Ric}_{j\bar{k}} = -3g_{j\bar{k}}$, i.e. the metric g_F is Kähler–Einstein with Einstein constant-3. By well-known results, Einstein metrics are real analytic (see e.g. [4]). Hence the metric g_F and the function F, defining $\mathcal{D}_F$, are real analytic in $(-x_0, x_0)$. The same is true for the function $G(x) = xF'^2 - FF' - xFF''$, namely the left hand side of equality (26). We claim that $G(x)$ is equal to a constant. Indeed, let $G(x) = \sum_{l=0}^{+\infty} b_l x^l$ be the converging Taylor expansion of $G(x)$ in a neighbourhood of

$x = 0$ and let $\Phi(z_1) = \sum_{j=0}^{+\infty} a_j z_1^j$ be the power expansion series of Φ around $0 \in D_{x_0}$. Then equality (26) translates as

$$\sum_{j,k=0}^{+\infty} a_j \bar{a}_k z_1^j \bar{z}_1^k = \sum_{l=0}^{+\infty} b_l |z_1|^{2l}.$$

Then all terms of the form $a_0\bar{a}_j$ with $j \neq 0$ are equal to zero. On the other hand $a_0 = \Phi(0) = -F(0)F^{'}(0) \neq 0$. This implies that $a_j = 0, \forall j > 1$ and hence, again by equality (26),

$$G(x) = xF'^2 - FF' - xFF'' = \Phi(0) = a_0, \tag{27}$$

which proves our claim. Taking the first derivative of (27) at zero one gets $2F(0)F''(0) = 0$. Since $F(0) \neq 0$, it follows that $F''(0) = 0$. Taking the higher order derivatives of (27) at zero one obtains

$$0 = \frac{\partial^k((F' + xF'')F - xF'^2)}{\partial x^k}(0) = (k+1)F(0)\frac{\partial^k F}{\partial x^k}(0), \quad k \geq 2,$$

and so $\partial^k F/\partial x^k(0) = 0$, $k \geq 2$. Using again the analyticity of F one obtains that $F(x) = \alpha - \beta x$, where α and β are positive constants. Then the map

$$\varphi : \mathcal{D}_F \to D^2 : (z_1, z_2) \mapsto \left(\sqrt{\frac{\beta}{\alpha}} z_1, \sqrt{\frac{1}{\alpha}} z_2\right)$$

is a biholomorphism satisfying

$$\varphi^*(\omega_{hyp}) = \omega_F$$

and this concludes the proof of our theorem. □

Remark 4.9. Observe that Theorem 4.8 is a generalization of a Theorem 3.1 in [14] where we prove that if g_F is Einstein then (D_F, ω_F) is biholomorphically isometric to (D_2, ω_{hyp}).

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